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# Hilbert irreducibility sequences and nonstandard arithmetic (Logic and the Foundations of Mathematics)

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CITATION:

Yasumoto, Masahiro. Hilbert irreducibility sequences and nonstandard arithmetic (Logic and the Foundations of Mathematics). 数理解析研究所講究録 1986, 588: 193-195

ISSUE DATE:

1986-04

URL:

<http://hdl.handle.net/2433/99414>

RIGHT:

# Hilbert irreducibility sequences and nonstandard arithmetic

(Masahiro Yasumoto)

Let  $\mathbb{Q}^*$  and  $\mathbb{Z}^*$  be enlargements of  $\mathbb{Q}$  and  $\mathbb{Z}$  respectively. Our aim of this paper is to give a sufficient condition for  $x \in \mathbb{Z}^* - \mathbb{Z}$  that  $\mathbb{Q}(x)$  has no algebraic extension of degree not more than  $m$  within  $\mathbb{Q}^*$ . As its application to number theory, we give irreducibility sequences explicitly.

By an arithmetical prime divisor, we mean a prime number or the archimedean prime  $p_\infty$ . For each arithmetical prime  $p$ , we define  $p$ -adic absolute value of a rational number  $x$ ,

$$|x|_p = p^{-n}$$

$$|x|_{p_\infty} = |x|$$

where  $x = rp^n$  and  $r$  has no  $p$  factor. For each finite set  $S$ , of arithmetical primes, we define

$$H_S(x) = \prod_{p \in S} \max(1, |x|_p)$$

$$H(x) = \prod_p \max(1, |x|_p) = \max(|m|, |n|)$$

where  $x = m/n$  and  $\text{g.c.d.}(m, n) = 1$ .

**THEOREM.** Let  $x$  be a nonstandard rational number. Assume  
 (1) there is a finite set  $S$  of standard prime divisors such that

$$\frac{\log(H_S(x)H_S(x^{-1}))}{\log H(x)} > 2 - \frac{1}{m} + \varepsilon$$

for some standard positive real  $\varepsilon$ ,

(2) for any nonzero standard rational number  $r$  and any natural number  $n$  with  $2 \leq n \leq m$ , there is no nonstandard rational  $y \in \mathbb{Q}^* - \mathbb{Q}$  such that  $rx = y^n$ .

Then  $\mathbb{Q}(x)$  has no algebraic extension of degree not more than  $m$  within  $\mathbb{Q}^*$ .

Let us give an application of the theorem to standard number theory. A sequence of integers  $a_1, a_2, \dots, a_n, \dots$  is called a  $m$ -irreducibility sequence if for any polynomial  $f(X, Y) \in \mathbb{Z}[X, Y]$  with  $\deg(f) \leq m$ , there are only finitely many  $a_n$  such that  $f(X, a_n)$  is reducible. A sequence of integers is called a Hilbert irreducibility sequence (H.i.seq.) if it is a  $m$ -irreducibility sequence for all natural number  $m$ . In his papers [3] and [4], V.G.Sprindzuk proved that

$$a_n = [\exp \sqrt{\log \log n}] + n! 2^{n^2}$$

is a H.i.seq.. Our theorem can give a different type of H.i. seq. from those given by Sprindzuk. For example, we will show that  $2^n p_n$ ,  $2^n(n^3+1)$  and  $n! 2^{n^2}$  are H.i.seq.s.

In nonstandard arithmetic, we have a beautiful characterization of a H.i.seq. due to Gilmor and Robinson.

**PROPOSITION 1.**  $a_n$  is a H.i.seq. if and only if for any nonstandard natural number  $\omega \in \mathbb{N}^* - \mathbb{N}$ ,  $\mathbb{Q}(a_\omega)$  is relatively

*algebraically closed in  $\mathbb{Q}^*$ .*

As for  $m$ -irreducibility we have the following sufficient condition for a sequence to be an  $m$ -irreducibility sequence

PROPOSITION 2. *If for any nonstandard natural number  $\omega$ ,  $\mathbb{Q}(a_\omega)$  has no proper algebraic extension of degree not more than  $m!$  within  $\mathbb{Q}^*$ , then  $a_n$  is an  $m$ -irreducibility sequence.*

Unfortunately the converse of Proposition 2 is not true but if  $m!$  is replaced by  $m$ , then its converse holds.

PROPOSITION 3. *If  $a_n$  is an  $m$ -irreducibility sequence, then for any nonstandard natural number  $\omega$ ,  $\mathbb{Q}(a_\omega)$  has no algebraic extension of degree not more than  $m$  within  $\mathbb{Q}^*$ .*

It is easily shown that Proposition 1 is a easy consequence of Proposition 2 and 3.

For the proofs of Theorem, Proposition 2 and 3, please refer to [5].

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